J. Fluid Mech. (1983), *vol.* 127, *pp.* 59–81
Printed in Great Britain

The evolution of Tollmien–Schlichting waves near a leading edge

By M. E. GOLDSTEIN
National Aeronautics and Space Administration, Lewis Research Center, Cleveland, Ohio 44135

(Received 23 November 1981 and in revised form 9 July 1982)

The method of matched asymptotic expansions is used to study the generation of Tollmien–Schlichting waves by free-stream disturbances incident on a flat-plate boundary layer. Near the leading edge, the motion is governed by the unsteady boundary-layer equation, while farther downstream it is governed (to lowest order) by the Orr–Sommerfeld equation with slowly varying coefficients. It is shown that there is an overlap domain where the Tollmien–Schlichting wave solutions to the Orr–Sommerfeld equation and appropriate asymptotic solutions of the unsteady boundary-layer equation match, in the matched-asymptotic-expansion sense. The analysis explains how long-wavelength free-stream disturbances can generate Tollmien–Schlichting waves of much shorter wavelength. It also leads to a set of scaling laws for the asymptotic structure of the unsteady boundary layer.

1. Introduction

It is well known that laminar to turbulent transition in boundary layers is strongly influenced by unsteady disturbances in the free stream. This is often the result of a sequence of events that begins with the excitation of spatially growing Tollmien–Schlichting waves by the free-stream disturbances. When the free-stream disturbances are periodic in time and of sufficiently small amplitude, the Tollmien–Schlichting waves will also be periodic.

This so-called receptivity† problem was discussed in a recent review article by Reshotko (1976). It differs from classical stability theory in that it leads to a boundary-value problem, while stability theory leads to an eigenvalue problem. Since the time-harmonic Tollmien–Schlichting waves are normal nodes of the Orr–Sommerfeld equation, which applies in the downstream region where the mean flow is nearly parallel, one can always add an arbitrary multiple of these waves to the solution of the boundary-value problem and still satisfy the boundary conditions and the governing equations unless an upstream boundary condition (i.e. an initial condition) is imposed at the start of the boundary layer. I.e. the Tollmien–Schlichting waves only couple to the free-stream disturbance when an upstream boundary condition is imposed.

However, this upstream boundary condition cannot be imposed on the solution to the Orr–Sommerfeld equation itself. Near the leading edge of the boundary layer (actually within a region that occupies the first few wavelengths of the boundary layer) the divergence of the mean flow has a first-order effect on the unsteady motion rather than being a higher-order effect that can be treated as a 'slowly varying' correction to a parallel flow as in classical stability theory. In this region, inertia terms involving the cross-stream component of the mean-flow velocity have to be included in the lowest-order equation for the unsteady flow. However, one can

† The term 'receptivity' was first introduced by Morkovin (1969).

neglect unsteady pressure fluctuations across the mean boundary layer, which is still relatively thin (on a wavelength scale). The flow is then governed by the unsteady boundary-layer equation rather than by an Orr–Sommerfeld equation with slowly varying coefficients.

This latter equation, whose eigensolutions are the Tollmien–Schlichting waves, is only valid further downstream. *The upstream boundary condition for its solutions should therefore be that they 'match', in the 'matched-asymptotic-expansion' sense* (*Cole* 1968) *onto the appropriate solutions of the unsteady boundary-layer equation in some intermediate region that overlaps the unsteady boundary layer and Orr–Sommerfeld regions.*

In order to reduce the problem to its simplest terms we restrict our attention to a two-dimensional incompressible time-stationary flow over an infinitely thin flat plate. The amplitude of the disturbance field is assumed to be small relative to the mean free-stream velocity U_∞ and the equations are linearized about the mean flow. Then since the unsteady flow is assumed to be time-stationary, only a single harmonic component of the disturbance field, say of frequency ω, need be considered. Even though we assume that the plate is infinitely thin, we ultimately show that our principal results apply to any flat plate whose 'nose radius' is of the order of the 'convective' wavelength U_∞/ω of the disturbance. The finite-thickness flat plate could have been considered at the outset, but this would have complicated the presentation. Finally, the Reynolds number based on U_∞/ω is assumed to be large.

Our approach is to take the reciprocal of this Reynolds number, which we denote by ϵ^6, as a small parameter and obtain a uniformly valid asymptotic expansion in this parameter. We suppose that the streamwise wavenumber of the imposed disturbance (i.e. its inverse spatial scale) is $O(\omega/U_\infty)$.† Then allowing $\epsilon \to 0$ while requiring that $x = \omega x^+/U_\infty$ be order one, where x^+ denotes the streamwise distance from the leading edge (see figure 1), one obtains the unsteady boundary-layer equation to lowest order of approximation (Moore 1951; Lighthill 1954).

This equation must be solved numerically, but one only needs to know the asymptotic expansion of its solution as $x \to \infty$ (i.e. far downstream) in order to show that it can be matched onto the Tollmien–Schlichting wave solution of the Orr–Sommerfeld equation. This asymptotic solution was studied by Lighthill (1954), Lam & Rott (1960) and Ackerberg & Phillips (1972). Their work shows that the solution develops a double-layer structure in this downstream region (actually this structure begins to develop when $x \approx 1$; see figure 1). The inner layer is a Stokes shear-wave type of flow to lowest order, and the outer flow is a modified Blasius motion. In fact, this flow will be identical with a Stokes shear wave to lowest order (Ackerberg & Phillips 1972) whenever the free-stream velocity fluctuation is asymptotically independent of x as $x \to \infty$.

Lam & Rott (1960) point out that the Stokes-type solution is essentially '*incomplete*' because it is uniquely determined independently of the upstream conditions that must always be imposed when solving a parabolic partial differential equation. The downstream unsteady boundary-layer solution therefore consists of the Stokes-layer-type solution supplemented by a set of asymptotic eigensolutions.

Lam & Rott (1960) went on to construct a set of such asymptotic eigensolutions, which decay exponentially fast in x, with the lowest-order eigensolution exhibiting the most rapid decay. Ackerberg & Phillips (1972) used the method of matched

† Note that this includes the zero-wavenumber disturbance corresponding to a uniform oscillation of the stream (or a plane acoustic wave in the incompressible limit).

asymptotic expansions to obtain expressions for these eigensolutions that are uniformly valid in η for $0 < \eta < \infty$, where η is the Blasius variable, i.e. it is the cross-stream coordinate divided by the local thickness of the steady boundary layer. However, it turns out that these expressions are not quite correct as they stand. Thus it is shown in §3 below that they still satisfy the unsteady boundary-layer equations to the same degree of approximation when they are multiplied by x^{τ} for any constant τ. But, it is also shown that there is only one value of the exponent τ for which the next-order equations can be solved, and this solvability condition uniquely (i.e. to within a constant factor) determines the lowest-order asymptotic eigensolutions. Brown & Stewartson (1973) found an alternative set of asymptotic eigensolutions of the unsteady boundary-layer equation whose exponential decay rate increases with increasing order.

The 'asymptotic eigensolutions' of Lam & Rott, which are proportional to $\exp(-\lambda_0 x^{\frac{3}{2}})$, where λ_0 is a complex constant, oscillate with a wavelength ($\sim x^{-\frac{1}{2}}$) that decreases with increasing x while the mean boundary-layer thickness increases at the same rate. Thus the spatial scale of the unsteady motion must ultimately become comparable to the boundary-layer thickness, and the cross-stream pressure fluctuations, which are neglected in the unsteady boundary-layer approximation, must then become important. The Lam & Rott eigensolutions, which are based on this approximation, will then be invalid (i.e. they will not be asymptotic solutions to the full Navier–Stokes equations).

We obtain new solutions, which apply further downstream than asymptotic eigensolutions of the unsteady boundary-layer equation, by using a generalization of the method of multiple scales (Nayfeh 1973, pp. 276–282) and considering the limiting form of the governing equation as $\epsilon \to 0$ with $x_1 \equiv \epsilon^2 x$ (rather than x) held fixed. This leads to solutions that apply when $x = O(\epsilon^{-2})$ (see figure 1). They are essentially the classical large-Reynolds-number–small-wavenumber approximation to the Tollmien–Schlichting wave solutions of the Orr–Sommerfeld equation, appropriately corrected for slow variation in boundary-layer thickness. Thus they decay exponentially fast in the downstream direction when x_1 is relatively small, and at least one of them exhibits exponential growth when x_1 is sufficiently large. One purpose of this paper is to show that there exists an overlap domain where these Tollmien–Schlichting waves match in the matched-asymptotic-expansion sense (Cole 1968) onto the Lam & Rott (1960) asymptotic eigensolutions, and therefore that the Tollmien–Schlichting waves are the natural continuations of these eigensolutions into the downstream region. Another purpose is to show that the classical solutions of the Orr–Sommerfeld equation at the lower branch of the neutral stability curve can be used to fix its spatial modes upstream of this curve.

The analysis explains how the long wavelength of the free-stream disturbance (assumed to be $O(U_\infty/\omega)$) is progressively reduced by non-parallel flow effects until it matches the Tollmien–Schlichting wavelength. A physical interpretation is given in §6. The amplitude of each Tollmien–Schlichting wave is equal to ϵ^s times a function whose order of magnitude is unity. Matching with the asymptotic eigensolutions allows us to determine the constant exponent s, i.e. to determine the dominant Reynolds-number scaling of the amplitude. The implications of this are discussed at the end of §6.

The Stokes-type solution of the boundary-layer equation remains uniformly valid everywhere in the downstream region and is completely decoupled from the Tollmien–Schlichting waves.

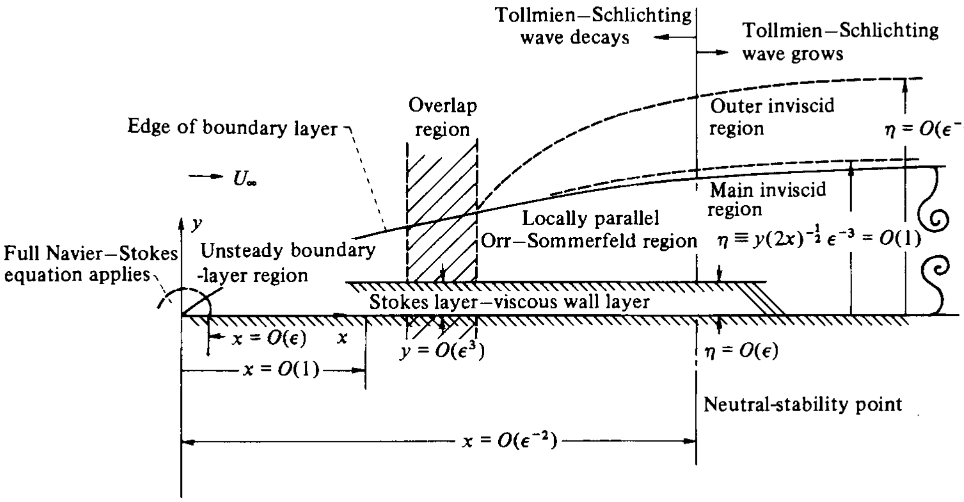

FIGURE 1. Asymptotic structure of unsteady boundary layer; $\epsilon = (\nu\omega/U_\alpha^2)^{\frac{1}{6}}$.

2. Formulation

We consider a two-dimensional incompressible flow of density ρ and kinematic viscosity ν over a semi-infinite flat plate as shown schematically in figure 1. Far upstream the motion consists of a uniform flow with velocity U_∞ plus a small-amplitude harmonic perturbation of frequency ω. We suppose that the velocity $\mathbf{v} = \{u, v\}$ has been non-dimensionalized by U_∞. We also suppose that the time t has been non-dimensionalized by ω^{-1} and that the Cartesian coordinates $\mathbf{x} = \{x, y\}$ have been non-dimensionalized by U_∞/ω. The plate is assumed to be located at $y = 0$, $x > 0$.

In the absence of viscosity the velocity at the surface of the plate would be of the form

$$u = 1 + u_\infty(x)\, e^{-it}, \quad v = 0 \quad (x > 0), \tag{2.1}$$

where u_∞ is assumed to be much less than unity. With viscosity, the motion is governed by the two-dimensional momentum and continuity equations which can be written in terms of the dimensionaless vorticity $-\Omega$ and stream function Ψ as

$$\frac{\partial\Omega}{\partial t} + \frac{\partial(\Omega, \Psi)}{\partial(x, y)} = \epsilon^6 \Delta\Omega, \tag{2.2}$$

$$\Omega = \Delta\Psi, \tag{2.3}$$

where $\partial(\Omega, \Psi)/\partial(x, y)$ denotes the Jacobian $(\partial\Omega/\partial x)(\partial\Psi/\partial y) - (\partial\Psi/\partial x)(\partial\Omega/\partial y)$, Δ denotes the Laplacian $(\partial^2/\partial x^2) + (\partial^2/\partial y^2)$, and

$$\epsilon^6 \equiv \nu\omega/U_\infty^2 \tag{2.4}$$

is the reciprocal of the characteristic Reynolds number of the problem. The velocity components are given by

$$u = \frac{\partial\Psi}{\partial y}, \quad v = -\frac{\partial\Psi}{\partial x}, \tag{2.5}$$

so that on the surface of the plate

$$\Psi = \frac{\partial\Psi}{\partial y} = 0, \quad y = 0 \quad (x > 0). \tag{2.6}$$

Our interest here is in the case where

$$\epsilon \ll 1, \tag{2.7}$$

$$\frac{du_\infty}{dx} = O(u_\infty), \tag{2.8}$$

$$x^{-1} = O(1).$$

Then x/ϵ^6, the Reynolds number based on the distance from the leading edge, will be large† and the viscous effects will be confined to a thin boundary layer at the surface of the plate. Hence the boundary condition at infinity is that Ψ match smoothly on to the inviscid solution at large values of y/ϵ^3. In fact in the absence of the perturbation u_∞ the steady flow is given by the extended Blasius series (Goldstein 1960, p. 142)

$$\Psi_{\mathrm{Bl}} = \epsilon^3\sqrt{2}\,\xi\{F(\eta)+O(\epsilon^6\xi^{-2}\ln(\xi\epsilon^{-3}))\}. \tag{2.9}$$

Here (ξ, η) are parabolic coordinates defined in the usual way by

$$z = x+iy = \chi^2, \tag{2.10}$$

$$\chi \equiv \xi+\frac{i\epsilon^3}{\sqrt{2}}\eta, \tag{2.11}$$

and F is the Blasius function, which is a solution of

$$F'''+FF'' = 0, \tag{2.12}$$

$$F(0) = F'(0) = 0, \tag{2.13}$$

$$F'(\eta) \to 1+\text{exponentially small terms} \quad \text{as} \quad \eta \to \infty, \tag{2.14}$$

where the prime denotes differentiation with respect to η. Even though ω appears in the non-dimensionalization of (2.2) and (2.3) it is clear that it does not appear in the Blasius solution (2.9).

Since we are interested in small-amplitude motion, it is natural to linearize the solution about the Blasius solution (2.9). We only retain the linear terms in this amplitude expansion, but it will be necessary to consider higher-order terms in the expansion in ϵ. However, these will all be of lower order than $\epsilon^6(\eta^2/\xi^2)$. It then follows from (2.9) that we can approximate the steady solution by the Blasius solution $\sqrt{2}\,\epsilon^3\xi F(\eta)$ and seek a solution of the form

$$\Psi = \epsilon^3[\sqrt{2}\,\xi F(\eta)+\psi(\xi^2,\eta)\,e^{-it}+\ldots], \tag{2.15}$$

where

$$|\psi| \ll |\xi F(\eta)|.$$

Moreover, (2.10) and (2.11) imply that

$$x = \xi^2\left[1+O\left(\frac{\epsilon^6\eta^2}{\xi^2}\right)\right], \tag{2.16}$$

$$\left|\frac{dz}{d\chi}\right|^2 = 4|\chi|^2 = 4\xi^2\left[1+O\left(\frac{\epsilon^6\eta^2}{\xi^2}\right)\right], \tag{2.17}$$

and that, to within this approximation,

$$\eta = y\epsilon^{-3}/(2x)^{\frac{1}{2}}. \tag{2.18}$$

† We are excluding a relatively small region near the leading edge, but we shall see that this need not concern us here.

Substituting (2.10), (2.11) and (2.15) into (2.2) and (2.3), subtracting out (2.12), and neglecting quadratic terms in ψ, we obtain upon using the approximation (2.17) to eliminate $|dz/d\chi|$ and the approximation (2.16) to reinsert the variable x in place of ξ,

$$-i\tilde{\Delta}\psi + x^{\frac{1}{2}}\left[\frac{\partial(x^{-1}\tilde{\Delta}\psi, x^{\frac{1}{2}}F)}{\partial(x,\eta)} + \frac{\partial(x^{-\frac{1}{2}}F'', \psi)}{\partial(x,\eta)}\right] = \tilde{\Delta}\left(\frac{1}{2x}\tilde{\Delta}\psi\right) + O(\psi\epsilon^6\Lambda) \quad (\eta, x > 0), \quad (2.19)$$

where

$$\tilde{\Delta} \equiv \frac{\partial^2}{\partial\eta^2} + 2\epsilon^6 x\frac{\partial^2}{\partial x^2} + \epsilon^6\frac{\partial}{\partial x}, \quad (2.20)$$

$$\Lambda \equiv \max\left\{\frac{\eta^2}{x}, x^{-1}\right\}.$$

This equation is sufficiently accurate to serve as a starting point for the present analysis. It must be solved subject to the boundary conditions

$$\psi = \frac{\partial\psi}{\partial\eta} = 0 \quad (\eta = 0, \quad x > 0),$$

since (2.18) shows that $\eta = 0$ on the surface of the plate. The solution to (2.19) must match on to the inviscid solution for large η.

3. Unsteady boundary-layer region

We first consider the limit $\epsilon \to 0$ with $x = O(1)$. With the present non-dimensionalization, this corresponds to letting the disturbance Reynolds number become infinite while keeping the streamwise distance at about a wavelength U_∞/ω from the leading edge. Then, in view of (2.8),

$$\psi = \psi_0(x,\eta) + O(\epsilon^6), \quad (3.1)$$

and ψ_0 satisfies

$$-i\psi_{0\eta\eta} + \frac{\partial^2}{\partial\eta\,\partial x}(F'\psi_{0\eta} - F''\psi_0) - \frac{1}{2x}\frac{\partial^2}{\partial\eta^2}(F\psi_{0\eta}) = \frac{1}{2x}\psi_{0\eta\eta\eta\eta}.$$

This equation can be integrated with respect to η to obtain the *linearized unsteady boundary-layer equation*

$$\left(-i + F'\frac{\partial}{\partial x}\right)\psi_{0\eta} - F''\frac{\partial\psi_0}{\partial x} - \frac{1}{2x}\frac{\partial}{\partial\eta}(F\psi_{0\eta}) - \frac{1}{2x}\psi_{0\eta\eta\eta} = h(x), \quad (3.2)$$

where $h(x)$ is determined by the free-stream pressure distribution. In fact, since this equation must be solved subject to the free-stream boundary condition (see (2.1), (2.5) and (2.15)).

$$u_1 \equiv \epsilon^3\psi_y = (2x)^{-\frac{1}{2}}\psi_{0\eta} \to u_\infty(x) + \text{exponentially small terms} \quad \text{as} \quad \eta \to \infty, \quad (3.3)$$

it follows from (2.14) that

$$h(x) = (2x)^{\frac{1}{2}}\left(\frac{\partial}{\partial x} - i\right)u_\infty(x). \quad (3.4)$$

On the surface of the plate ψ_0 satisfies

$$\psi_0 = \psi_{0\eta} = 0 \quad (\eta = 0). \quad (3.5)$$

Equation (3.2), being the boundary-layer approximation to the Navier–Stokes equation, neglects pressure variations across the layer but accounts for the divergence or non-parallelism of the mean flow in that it retains inertia terms, like $V(\partial^2\psi/\partial y^2)$, where V is the mean cross-stream velocity.

As we indicated in §1, our interest is in showing that certain asymptotic solutions to this equation, which applies when $x = O(1)$, can match in the 'matched-asymptotic-expansion' sense (Cole 1968) onto the Tollmien–Schlichting wave solutions of the Orr–Sommerfeld equation as x becomes large.

We consider only the asymptotic eigensolutions of Lam & Rott (1960). Their work precedes the use of matched asymptotic expansions, but Ackerberg & Phillips rederived their results using this approach. We refer mainly to this more modern work,† which clearly shows that the asymptotic eigensolutions provide asymptotic solutions to the unsteady boundary-layer equation which are uniformly valid in η for $0 < \eta < \infty$ and exhibit a two-layer structure with adjustment to the wall boundary conditions taking place across a thin inner layer of the same thickness as the Stokes layer.

In fact, in the present notation, their formula for the asymptotic eigenfunctions becomes‡

$$\psi_{\mathrm{AP}} = C g_0(x,\eta)\exp\left[-\frac{\lambda(2x)^{\frac{3}{2}}}{3U_0'}\right] \quad \text{as} \quad x \to \infty, \tag{3.6}$$

where C is an arbitrary constant, and

$$g_0 = \begin{cases} \dfrac{iU_0'}{\lambda} + (2x)^{\frac{1}{2}}F'(\eta) + O(x^{-\frac{3}{2}}) & \text{in the main boundary layer,} \quad \eta = O(1), \quad (3.7a) \\[2ex] \dfrac{U_0'\displaystyle\int_0^{\sigma}(\sigma-\tilde{\sigma})\,w(\tilde{\sigma})\,d\tilde{\sigma}}{\displaystyle\int_0^{\infty} w(\tilde{\sigma})\,d\tilde{\sigma}} + O(x^{-\frac{3}{2}}) & \text{for} \quad \eta = O(x^{-\frac{1}{2}}). \quad (3.7b) \end{cases}$$

F' is defined by (2.12)–(2.14),§

$$U_0' \equiv F''(0) = 0{\cdot}4696\ldots; \tag{3.8}$$

$$\lambda \equiv e^{-\frac{7}{4}\pi i}/\zeta_n^{\frac{3}{2}}, \tag{3.9}$$

$$w(\sigma) \equiv \mathrm{Ai}\,(\zeta_{\mathrm{b}}), \tag{3.10}$$

$$\zeta_{\mathrm{b}} \equiv (1 - i\sigma\lambda)\,\zeta_n, \tag{3.11}$$

$$\sigma = (2x)^{\frac{1}{2}}\eta = y/\epsilon^3, \tag{3.12}$$

and ζ_n denotes the nth root of

$$\mathrm{Ai}'(\zeta_n) = 0 \quad (n = 1, 2, 3, \ldots). \tag{3.13}$$

Here Ai and Ai′ denote the Airy function and its derivative in the usual notation (Abramowitz & Stegun 1964, pp. 446, 448). Since the roots of (3.13) all lie along the negative real axis, we can put

$$\zeta_n = \rho_n e^{-i\pi} \quad \text{with} \quad \rho_n > 0. \tag{3.14}$$

It is easy to see that (3.6) oscillates with increasing rapidity (i.e. the wavelength of the oscillation decreases like $x^{-\frac{1}{2}}$) as $x \to \infty$.

Ackerberg & Phillips (1972) show that this result is a homogeneous solution of the

† The Ackerberg & Phillips analysis was restricted to the case of constant u_∞, but the asymptotic eigensolutions do not depend on u_∞ and therefore remain unchanged in the more general case considered herein. Of course the arbitrary constants that multiply these eigensolutions will be strongly dependent on the precise nature of $u_\infty(x)$.

‡ It is important to note that Ackerberg & Phillips use an e^{it} time dependence while we use e^{-it}, so our results are essentially the complex conjugates of theirs.

§ Note that we set $\beta = +\frac{2}{3}\pi$ rather than $-\frac{2}{3}\pi$ as was done by Ackerberg & Phillips.

unsteady boundary-layer equation (3.2)‖ to within an error that is smaller than (3.6) by a factor $O(x^{-\frac{3}{2}})$. However, as shown in appendix A, $\psi_c = x^{\tau}\psi_{AP}$ also provides a solution with this property for any constant τ. The exponent τ is determined by the higher-order terms.

One way to find this quantity is to begin with an expansion of the form

$$\psi_c = x^{\tau}(g_0 + x^{-\frac{3}{2}}g_1 + \ldots)\exp\left[-\frac{\lambda(2x)^{\frac{3}{2}}}{3U_0'}\right] \quad \text{as} \quad x \to \infty, \tag{3.15}$$

where g_0 is given by (3.7), and show that there is only one value of τ that leads to an equation for g_1 that possesses a solution (Nayfeh 1973, pp. 276–282). This procedure is carried out in appendix A, where it is shown that τ is given by

$$\tau = -\frac{\int_0^{\infty}[\sigma^2 g_0'' - 2\sigma g_0' + \frac{1}{3}\lambda\sigma^3(\frac{1}{4}\sigma g_0' - g_0)]\,w'\,d\sigma}{4\int_0^{\infty}(\sigma g_0' - g_0)\,w'\,d\sigma}, \tag{3.16}$$

and g_0 and w are given by (3.7b) and (3.10) respectively.

4. The Orr–Sommerfeld region

Since $\arg\lambda = -\frac{1}{4}\pi$, (3.15) represents a 'downstream-travelling' disturbance whose wavelength is decreasing like $x^{-\frac{1}{2}}$. Then the cross-stream velocity perturbation

$$v_1 \equiv -\epsilon^3\frac{\partial\psi}{\partial x} \tag{4.1}$$

will eventually become large relative to the streamwise velocity perturbation

$$u_1 = \epsilon^3\frac{\partial\psi}{\partial y}. \tag{4.2}$$

This produces a significant cross-section pressure fluctuation through the transverse momentum equation, and the unsteady boundary-layer equation (3.2), from which (3.15) is derived, becomes invalid. In fact, substituting (3.15) in $\tilde{\Delta}\psi$, where $\tilde{\Delta}$ is defined by (2.20), shows that the second term in the resulting expression will eventually be proportional to $(2\lambda x\epsilon^3/U_0')^2$, and therefore when $x = O(\epsilon^{-3})$ it will certainly not be negligible compared with the first term, as was assumed in deriving the unsteady boundary-layer equation (3.2). However, it turns out that (3.15) breaks down at even smaller values of x. This occurs because $\partial u_1/\partial y = \epsilon^3\partial^2\psi/\partial y^2$ is small in the outer portion of the boundary layer and $\partial v_1/\partial x = -\epsilon^3\partial^2\psi/\partial x^2$ therefore becomes significant at smaller values of x there.

We therefore seek a solution to (2.19) that is valid in the region where

$$x_1 = \epsilon^r x \quad (0 < r \leqslant 3) \tag{4.3}$$

remains of order unity as $\epsilon \to 0$ and which extends the asymptotic unsteady boundary-layer solution (3.15) into this region, that is, which matches the latter solution asymptotically as $x_1 \to 0$.

The character of (3.15) suggests that this solution will be of the form

$$\psi = \epsilon^s G(x_1, \eta, \epsilon)\exp\left[\frac{i}{\epsilon^b}\int_0^x \kappa(x_1, \epsilon)\,dx\right], \tag{4.4}$$

where κ and G are $O(1)$,
$$b \equiv \tfrac{1}{2}r, \tag{4.5}$$

and the constant s will be determined by the analysis.

‖ Our dependent variable differs from that of Ackerberg & Phillips by a factor of $x^{\frac{1}{2}}$.

Substituting (4.4) into (2.19), we obtain

$$\frac{1}{\alpha R}\mathscr{L}G = -\frac{\epsilon^{\frac{3}{2}r}}{\kappa}\Big\{U(3\alpha\alpha_{x_1}G+2\alpha^2G_{x_1})-c(\alpha\alpha_{x_1}G+2\alpha^2G_{x_1})+U''G_{x_1}$$
$$-U(D^2-\alpha^2)\,G_{x_1}+\frac{U}{x_1}(D^2-\alpha^2)\,G+\frac{F}{2x_1}(D^2-\alpha^2)\,DG+\frac{U'}{2x_1}DG\Big\}+O(\epsilon^{6+r})$$

uniformly in η. (4.6)

Here $D \equiv \partial/\partial\eta$, the primes denote total derivatives with respect to η, the subscript x_1 denotes partial or total derivatives with respect to x_1,

$$R = R(x_1) \equiv \frac{(2x)^{\frac{1}{2}}}{\epsilon^3} = \left(\frac{2x_1}{\epsilon^{r+6}}\right)^{\frac{1}{2}}, \tag{4.7}$$

$$\alpha = \alpha(x_1) \equiv \epsilon^{3-r}(2x_1)^{\frac{1}{2}}\kappa(x_1), \tag{4.8}$$

$$c = c(x_1) = \frac{\epsilon^{\frac{1}{2}r}}{\kappa(x_1)}, \tag{4.9}$$

$\mathscr{L}$ denotes the Orr–Sommerfeld operator

$$\mathscr{L} \equiv (D^2-\alpha^2)^2 - i\alpha R[(U-c)(D^2-\alpha^2)-U''], \tag{4.10}$$

and, as can be seen from (2.5), (2.15), (2.16) and (2.18),

$$U = U(\eta) \equiv F'(\eta) \tag{4.11}$$

is the mean-flow velocity in the direction along the plate.

The statement at the end of (4.6) is intended to imply that we have retained sufficient terms to ensure that the approximation will be valid for all η, even in local regions where the terms involving higher derivatives with respect to η can become large and in regions where U and/or its derivatives become small (i.e. near the wall and at the outer edge of the Blasius boundary layer). Notice, for example, that, even though the effective Reynolds number R will always be large and α and/or c will always be small, we have included terms involving these quantities on the left-hand side of (4.6), which presumably contains only the lowest-order terms.

The left side of (4.6) is then, just the Orr–Sommerfeld equation with coefficients α, c and R that are slowly varying functions of x, that is, they are functions of the 'slow variable' x_1. However, α, c and R are not all independent but, as can be seen from (4.7)–(4.9), are related to each other and to x_1 by

$$\alpha c = (2x_1)^{\frac{1}{2}}\epsilon^{3-\frac{1}{2}r} = \epsilon^6 R. \tag{4.12}$$

Of course α and c correspond respectively to the wavenumber and wave speed in (4.10).

Since (4.6) has slowly varying coefficients, it is appropriate (Saric & Nayfeh 1975) to put

$$G = A(x_1)\,\gamma(\eta, x_1), \tag{4.13}$$

where A is a 'slowly varying' function of x_1 to be determined by the analysis. Then (4.6) becomes

$$\frac{1}{\alpha R}\mathscr{L}\gamma = -\frac{\epsilon^{\frac{3}{2}r}}{\kappa}\left(H_1\frac{d\ln A}{dx_1}+H_2\right)+O(\epsilon^{6+r}), \tag{4.14}$$

$$H_1 = [\alpha^2(3U-2c)+U'']\,\gamma - UD^2\gamma, \tag{4.15}$$

$$H_2 = \alpha\alpha_{x_1}(3U-c)\,\gamma+[U''+\alpha^2(3U-2c)-UD^2]\,\gamma_{x_1}+\frac{1}{2x_1}\left[D^2(FD\gamma)-\frac{\alpha^2}{F}D(F^2\gamma)\right]. \tag{4.16}$$

It follows from (3.5), (4.4) and (4.13) that we must require

$$\gamma = D\gamma = 0 \quad \text{at} \quad \eta = 0 \tag{4.17}$$

and it is appropriate to require that

$$D\gamma \to 0 \quad \text{(exponentially)} \quad \text{as } \eta \to \infty. \tag{4.18}$$

Since the effective Reynolds number R is always large in the present approximation, it is only appropriate to consider the asymptotic (as $R \to \infty$) solution to (4.14). Fortunately, the asymptotic theory of the Orr–Sommerfeld equation has been highly developed by Tollmien (1929, 1947), Lin (1945, 1955) and many others.

As we indicated in §1, our interest is in showing that (4.14) has an eigensolution (i.e. a solution satisfying the homogeneous boundary conditions (4.17) and (4.18)) that matches the damped asymptotic eigensolution (3.15) when x_1 is small and develops into a growing (i.e. unstable) wave when $x_1 = O(1)$.

Now it can be seen from (3.15), (4.4) and (4.8) that matching with (3.15) can only occur if $\alpha \sim x_1 \epsilon^{3-r}$ as $x_1 \to 0$, which, in view of (4.3), implies that α must be small in this limit. But it is also known (Reid 1965, p. 306; Lin 1946) that α is small in the vicinity of the neutral stability curve when R is large. Hence it is appropriate to restrict our attention to the case where α is small.

However, it can be seen from (4.10) and (4.14) that R actually appears in the combination αR in the Orr–Sommerfeld equation, and the asymptotic theory for large R and small α must really be an asymptotic theory for large αR and small α. But it follows from (4.7) and (4.8) that

$$\alpha R = 2x_1 \kappa(x_1, \epsilon)/\epsilon^{\frac{3}{2}r} \tag{4.19}$$

will certainly be large in the present case.

It remains to choose the scaling exponent r. As we have already indicated, we would like to do this so that x_1 will be of order unity in the vicinity of the lower branch of the neutral-stability curve. The asymptotic theory of the Orr–Sommerfeld equation for large αR and small α (Reid 1965, pp. 279–281; Lin 1946) shows that $\alpha = O(c)$ in this region. Hence it follows from (4.8) and (4.9) that we must put

$$r = 2. \tag{4.20}$$

Then

$$\alpha = \epsilon\bar{\alpha}, \quad c = \epsilon\bar{c}, \tag{4.21}$$

where

$$\bar{\alpha} = (2x_1)^{\frac{1}{2}}\kappa \tag{4.22}$$

and

$$\bar{c} = \kappa^{-1} \tag{4.23}$$

are $O(1)$.

It is important to notice that, even though κ and consequently $\bar{\alpha}$ and $\bar{c}$ are $O(1)$, κ *does* depend on ϵ and therefore possesses an asymptotic expansion in this parameter. However, it is clear from (4.3)–(4.5) and (4.20) that this expansion need only be carried out up to, but not including, terms $O(\epsilon^4)$, since (see (4.3), (4.5) and (4.20)) the latter can always be incorporated into the 'slowly varying' function $A(x_1)$, which enters (4.4) via (4.13). Thus, in the present approach, the slowly varying amplitude function, which is usually introduced to account for non-parallel-flow effects (Gaster 1974; Saric & Nayfeh 1975), is merely the natural continuation of the asymptotic expansion of κ.

Since α and c are both small, classical theory (Lin 1946) suggests that the solution will exhibit a three-layer structure in the η-direction. There will be (i) a viscous wall layer that contains the critical layer, (ii) a main inviscid layer where the flow is quasi-steady and nearly parallel, since α and c appear only in the higher-order terms there, and (iii) an outer inviscid region where unsteady effects and streamwise variations are important since α enters the lowest-order solution in this region.

Smith (1979) recognized that the Tollmien–Schlichting waves exhibit this triple-layer structure on the lower branch of the neutral-stability curve and showed that this structure is, in fact, the same as the 'triple-deck' structure of Stewartson (1969) and Messiter (1970) there. In Smith's analysis, the solution is expanded in terms of the usual triple-deck parameter $\epsilon^* = (\nu/U_\infty x^+)^{\frac{1}{8}}$, which, as is easily seen from (2.4), (4.3) and (4.20), is related to our small parameter ϵ by $\epsilon = \epsilon^* x_1^{\frac{1}{4}}$. Hence both parameters are of the same order in the outer region where $x_1 = O(1)$ and we should be able to obtain the solution to the present problem by using a procedure close to the one used by Smith (1979). But, since Smith's parameter depends on x while ours does not, we do not have to expand the frequency in powers of ϵ as was done by Smith. Our procedure turns out to be intermediate between that used by Smith, and that of classical Tollmien–Schlichting theory. We therefore omit the details and only outline the major steps. We first consider the inviscid layers.

4.1. *The inviscid region*

The solution in the main inviscid layer corresponds to the limit $\epsilon \to 0$ with $\eta = O(1)$. We therefore seek an expansion of the form

$$\gamma = \gamma_0(\eta, x_1) + \epsilon\gamma_1(\eta, x_1) + O(\epsilon^2). \tag{4.24}$$

Since κ, and consequently $\bar{\alpha}$ and $\bar{c}$, depend on ϵ, the conventional approach would be to also expand these quantities in ϵ (Nayfeh 1973, pp. 68–71), but it turns out to be simpler to leave them unexpanded and determine their ϵ-dependence at the end of the analysis. This introduces extraneous higher-order terms into the expansions, but of course no additional error is incurred by retaining such terms. They can be eliminated at any stage of the analysis by re-expanding the solution – if this turns out to be desirable. Terms such as γ_0 and γ_1 will then depend on ϵ as well as the indicated arguments but we simplify the notation by suppressing this dependence. Another reason for leaving $\bar{\alpha}$ and $\bar{c}$ unexpanded is to keep the analysis as close as possible to classical stability theory.

Matching with the wall-layer solution requires that the lowest-order normal velocity component of the outer solution vanish at the wall – so we must take

$$\gamma_0(0, x_1) = 0. \tag{4.25}$$

The solution in the outer inviscid region corresponds to the limit $\epsilon \to 0$ with $\tilde{\eta} = O(1)$, where

$$\tilde{\eta} = \epsilon\eta. \tag{4.26}$$

In this region the expansion must be of the form

$$\gamma = \tilde{\gamma}_0(\tilde{\eta}, x_1) + \epsilon\tilde{\gamma}_1(\tilde{\eta}, x_1) + O(\epsilon^2). \tag{4.27}$$

When the condition (4.25) is imposed, the first two terms of these two expansions can be determined independently of the solution in the wall layer. Since the procedure is standard and is in fact the same as that used by Smith (1979), we only give the result, which in the main inviscid layer is

$$\gamma = U - (\bar{c}\epsilon) - \epsilon\bar{\alpha}U\left[\int_\infty^\eta \left(\frac{1}{U^2} - 1\right) d\eta + \eta\right] + O(\epsilon^2). \tag{4.28}$$

The analysis can easily be continued to determine the higher-order terms, which up to, but not including, those of $O(\epsilon^5)$ are found to satisfy Rayleigh's equation with $\alpha = \epsilon\bar{\alpha}$ and $c = \epsilon\bar{c}$ both small. A number of investigators obtained uniformly valid asymptotic solutions to this equation for the limit $\alpha \equiv \epsilon\bar{\alpha} \to 0$. One such solution that satisfies the appropriate outer boundary condition is given by Reid (1965, p. 279). The solution to the present problem is easily obtained by re-expanding his result for small values of $c = \epsilon\bar{c}$ (see Goldstein 1982).

4.2. *The viscous wall layer and the characteristic equation*

The solution in the viscous wall layer corresponds to the limit $\epsilon \to 0$, $\bar{\eta} = O(1)$, where

$$\bar{\eta} \equiv \eta/\epsilon. \tag{4.29}$$

We therefore seek an expansion of the form

$$\gamma = \epsilon b(x_1, \epsilon)\,\bar{\gamma}_0(\bar{\eta}, x_1) + \epsilon^4 \bar{\gamma}_4(\bar{\eta}, x_1) + \ldots, \tag{4.30}$$

where $$b(x_1, \epsilon) = 1 + \epsilon b_1(x_1) + \epsilon^2 b_2(x_1) + \epsilon^3 (\ln \epsilon)\, b_3(x_1).$$

Equation (4.19) and (4.20) show that

$$\bar{\beta} \equiv \epsilon(\alpha R)^{\frac{1}{3}} = (2x_1 \kappa)^{\frac{1}{3}} \tag{4.31}$$

is of order unity. Inserting (4.20), (4.21) and (4.29)–(4.31) into (4.14)–(4.16), using (4.11) and (A 5), and equating coefficients of like powers of ϵ yields

$$\mathscr{L}_{\mathrm{w}} \bar{\gamma}_0 = 0, \tag{4.32}$$

$$\mathscr{L}_{\mathrm{w}} \bar{\gamma}_4 = -U_0' \bar{c} \bar{\beta}^3 \left(\bar{H}_1 \frac{d \ln A}{dx_1} + \bar{H}_2 \right), \tag{4.33}$$

where the operator $\mathscr{L}_{\mathrm{w}}$ is defined by

$$\mathscr{L}_{\mathrm{w}} \equiv \bar{D}^4 - i\bar{c}\bar{\beta}^3 \left(\frac{U_0' \bar{\eta}}{\bar{c}} - 1 \right) \bar{D}^2, \quad \bar{D} \equiv \frac{d}{d\bar{\eta}}, \tag{4.34}$$

and the functions $\bar{H}_1$ and $\bar{H}_2$ are defined by

$$\bar{H}_1 \equiv \bar{D}(\bar{\gamma}_0 - \bar{\eta}\bar{D}\bar{\gamma}_0), \quad \bar{H}_2 \equiv \bar{D}\bar{H}, \tag{4.35}$$

where $$\bar{H} \equiv \frac{\partial}{\partial x_1}(\bar{\gamma}_0 - \bar{\eta}\bar{D}\bar{\gamma}_0) + \frac{1}{4x_1} \bar{D}(\bar{\eta}^2 \bar{D}\bar{\gamma}_0) - \frac{iU_0'}{\bar{c}\, 3!} \bar{\eta}^3 (\bar{\gamma}_0 - \tfrac{1}{4}\bar{\eta}\bar{D}\bar{\gamma}_0). \tag{4.36}$$

The boundary condition (4.17) implies that

$$\bar{\gamma}_0 = \bar{D}\bar{\gamma}_0 = 0 \quad \text{at} \quad \bar{\eta} = 0, \tag{4.37}$$

$$\bar{\gamma}_4 = \bar{D}\bar{\gamma}_4 = 0 \quad \text{at} \quad \bar{\eta} = 0. \tag{4.38}$$

Equation (4.32) was used by Lin (1945) to describe the Tollmien–Schlichting waves in the neighbourhood of the critical layer. Its solution is well known. In fact introducing the new independent variable

$$\zeta \equiv \zeta_0 \left(1 - \frac{U_0' \bar{\eta}}{\bar{c}} \right), \tag{4.39}$$

where $$\zeta_0 \equiv \frac{e^{-\frac{5}{6}\pi i} \bar{c} \bar{\beta}}{(U_0')^{\frac{2}{3}}} = \frac{e^{-\frac{5}{6}\pi i} (\alpha R U_0')^{\frac{1}{3}} c}{U_0'}, \tag{4.40}$$

into this result shows that $d^2\bar{\gamma}_0/d\zeta^2$ satisfies Airy's equation

$$\frac{d^2\,\mathrm{Ai}}{d\zeta^2}-\zeta\,\mathrm{Ai}=0. \tag{4.41}$$

The resulting solution for $\bar{\gamma}_0$ is given in appendix B.

The inner expansion (4.30) must now be matched onto the solution in the main inviscid region. This will ultimately determine the expansion of κ in terms of ϵ that was alluded to above.

Since the procedure is again similar to that used by Smith (1979), and is roughly equivalent to the now classical procedure of Lin (1945, 1946), we give only the final result. The analysis shows that the inner and outer expansions will match to within terms of $O(\epsilon^4\ln\epsilon)$ if we put (for details see Goldstein 1982)

$$1-\frac{\bar{\alpha}}{\bar{c}U_0'}\left[1-\epsilon\left(2\bar{c}-\frac{\bar{\alpha}}{U_0'}J_1\right)+\epsilon^2\left(\bar{c}^2+\frac{2\bar{\alpha}\bar{c}}{U_0'}J_2+\frac{\bar{\alpha}^2}{U_0'^2}J_3\right)+\tfrac{1}{2}\epsilon^3\frac{\bar{\alpha}\bar{c}^2}{U_0'^3}\ln\left(-\frac{\epsilon\bar{c}}{U_0'}\right)\right]=F^+(\zeta_0), \tag{4.42}$$

where

$$F^+(\zeta_0)\equiv\frac{\int_{\infty_1}^{\zeta_0}d\zeta\int_{\infty_1}^{\zeta}\mathrm{Ai}(\tilde{\zeta})\,d\tilde{\zeta}}{\zeta_0\int_{\infty_1}^{\zeta_0}\mathrm{Ai}(\zeta)\,d\zeta} \tag{4.43}$$

is the Tietjens function, and J_1, J_2, J_3 are constants defined by

$$J_1\equiv U_0'\int_0^\infty\left(U^2-\frac{1}{U^2}+\frac{1}{U_0'^2\eta^2}\right)d\eta, \tag{4.44}$$

$$J_2=-U_0'\int_0^\infty\left(\frac{1}{U^3}-\frac{2}{U^2}+U-\frac{1}{(U_0'\eta)^3}+\frac{2}{(U_0'\eta)^2}\right)d\eta, \tag{4.45}$$

$$J_3=J_1^2-2U_0'^2\int_0^\infty U^2\int_\eta^\infty\left(U^2-\frac{1}{U^2}\right)d\eta\,d\eta.$$

Not surprisingly, this is precisely the characteristic equation that is obtained from the classical large-αR small-α asymptotic solution to the Orr–Sommerfeld equation with the irrelevant higher-order terms in c neglected (Lin 1946, p. 294 of appendix and equation immediately following (12.5);† Reid 1965, pp. 279–282). It applies when $c=O(\alpha)$. Its solutions are the eigenvalues of the Orr–Sommerfeld equation associated with the Tollmien–Schlichting instability waves.

The neutral-stability curve, which divides the region of growing instability waves from the region of decaying waves, corresponds to real values of α and c. The solution to (4.42) corresponding to the lower branch of this curve is given to lowest order in α and c (i.e. to lowest order in ϵ) by (Lin 1946, equation (12.7); Reid 1965, pp. 281–282):

$$c\approx 2{\cdot}296\alpha/U_0', \tag{4.46}$$

$$R\approx 1{\cdot}002(U_0')^5/\alpha^4. \tag{4.47}$$

The first of these shows that, as was anticipated, α is indeed of order c in the vicinity of this curve, while the second shows that αR is large there.

The difference between the present result and that of conventional stability theory is that α, c and R are no longer independent, but are related to each other and to

† There are some minor typographical errors in equation (7) of Lin's appendix and a prime is missing in his equation (12.5).

x_1 by (4.7)–(4.9) with $r = 2$ (see also (4.12)). Then since only κ and x_1 appear in the actual solution (4.4), it will be helpful to eliminate α, c and R and express the characteristic equation (4.42) entirely in terms of κ and x_1. To this end we integrate by parts to obtain

$$\int_{\infty_1}^{\zeta} d\zeta \int_{\infty_1}^{\zeta} \mathrm{Ai}\,(\bar{\zeta})\, d\bar{\zeta} = \zeta \int_{\infty_1}^{\zeta} \mathrm{Ai}\,(\zeta)\, d\zeta - \int_{\infty_1}^{\zeta} \zeta\, \mathrm{Ai}\,(\zeta)\, d\zeta. \tag{4.48}$$

Hence it follows from (4.41) that (4.43) can be written as

$$F^{+}(\zeta_0) = 1 - \frac{\mathrm{Ai}'\,(\zeta_0)}{\zeta_0 \displaystyle\int_{\infty_1}^{\zeta_0} \mathrm{Ai}\,(\zeta)\, d\zeta}. \tag{4.49}$$

On the other hand, it follows from (4.23) and (4.31) that (4.40) can be written as

$$\zeta_0 = e^{-\frac{5}{6}\pi i}(\tilde{x}_1^{\frac{1}{2}}/\kappa)^{\frac{2}{3}}, \tag{4.50}$$

where

$$\tilde{x}_1 \equiv 2x_1/U_0'^2. \tag{4.51}$$

Inserting (4.22), (4.23), (4.49) and (4.50) into (4.42) yields

$$\tilde{x}_1^{\frac{3}{2}} + (\epsilon\, e^{\frac{1}{4}i\pi}\, \zeta_0^{\frac{3}{2}})\, \tilde{x}_1 \left(2 - \frac{\tilde{x}_1^{\frac{3}{2}} J_1}{i\zeta_0^3}\right) + (\epsilon\, e^{\frac{1}{4}i\pi}\, \zeta_0^{\frac{3}{2}})^2\, \tilde{x}_1^{\frac{1}{2}} \left(1 + \frac{2\tilde{x}_1^{\frac{3}{2}} J_2}{i\zeta_0^3} - \frac{\tilde{x}_1^3 J_3}{\zeta_0^6}\right)$$
$$- \frac{e^{\frac{1}{4}i\pi}(\tilde{x}_1 \zeta_0)^{\frac{3}{2}}\epsilon^3}{2U_0'^2} \ln\left(\frac{\epsilon\, e^{\frac{1}{4}i\pi}\, \zeta_0^{\frac{3}{2}}}{\tilde{x}_1^{\frac{1}{2}}\, U_0'}\right) = H(\zeta_0) \equiv \frac{e^{\frac{5}{2}i\pi}\zeta_0^2\, \mathrm{Ai}'\,(\zeta_0)}{\displaystyle\int_{\infty_1}^{\zeta_0} \mathrm{Ai}\,(\zeta)\, d\zeta}. \tag{4.52}$$

These equations determine the exponent κ in the solution (4.4) as a function of the 'slow variable' x_1 and the small parameter ϵ. They are accurate to within an error $O(\epsilon^4)$, but, as we have already indicated, there is no need to determine κ with any greater precision because the higher-order effects can now be accounted for by the slowly varying amplitude function $A(x_1)$. The latter is easily obtained by extending the analysis to include the $O(\epsilon^5)$ terms in the outer expansion and the $O(\epsilon^4)$ terms in the inner, but we choose to terminate the expansion at the present order.

5. Matching of asymptotic eigensolution and Tollmien–Schlichting wave

We must now show that as $x_1 \equiv \epsilon^2 x \to 0$, the downstream solution (4.4), which applies when $x_1 = O(1)$, matches the asymptotic eigensolution (3.15), of the unsteady boundary-layer equation, which applies in the region where $x = O(1)$. To this end, we first show that the exponential term in (4.4) matches the exponential term in (3.15). The function γ, which enters through (4.13), will then be shown to match $g_0 x^{-\frac{1}{2}}$. Then the Tollmien–Schlichting wave (4.4) will completely match the asymptotic eigensolution (3.15) if $A(x_1) \sim x_1^{\tau+\frac{1}{2}}$ as $x_1 \to 0$ provided that we set the exponent s equal to $-(2\tau+1)$. This last step is straightforward, but rather involved, and we do not give the details here. The interested reader is referred to Goldstein (1982).

5.1. *Matching of exponential terms*

As we have already indicated, (4.50) and (4.52), which are only accurate to $O(\epsilon^4)$, contain irrelevant higher-order terms owing to their implicit dependence on ϵ through κ. We can eliminate these terms by expanding κ in an appropriate asymptotic series in ϵ and then re-expanding the result.

Since H is an analytic function of ζ_0, it is clear that κ must have an expansion of the form

$$\kappa = \kappa_0 + \epsilon\kappa_1 + \epsilon^2\kappa_2 + \epsilon^3(\ln \epsilon)\,\kappa_3 + O(\epsilon^3). \tag{5.1}$$

Inserting this into (4.50) and (4.51), expanding H in a Taylor series about

$$\zeta_{00} \equiv e^{-\frac{5}{6}\pi i}\left(\frac{\tilde{x}_1^{\frac{1}{2}}}{\kappa_0}\right)^{\frac{2}{3}}, \tag{5.2}$$

and equating coefficients of like orders of ϵ, we obtain

$$H(\zeta_{00}) = \tilde{x}_1^{\frac{3}{2}}, \tag{5.3}$$

$$\frac{\kappa_1}{\kappa_0} = -\tfrac{3}{2}e^{\frac{1}{4}i\pi}\zeta_{00}^{\frac{1}{2}}\tilde{x}_1\left(2-\frac{\tilde{x}_1^{\frac{3}{2}}J_1}{i\zeta_{00}^3}\right)\Big/H'(\zeta_{00}), \tag{5.4}$$

$$\frac{\kappa_2}{\kappa_0} = -\frac{1}{3}\left[\frac{1}{2}-\frac{H''(\zeta_{00})\,\zeta_{00}}{H'(\zeta_{00})}\right]\left(\frac{\kappa_1}{\kappa_0}\right)^2 + 3e^{-\frac{1}{4}i\pi}\left(\frac{\tilde{x}_1}{\zeta_{00}}\right)^{\frac{5}{2}}J_1\left(\frac{\kappa_1}{\kappa_0}\right)\Big/H'(\zeta_{00})$$
$$-\tfrac{3}{2}i\zeta_{00}^2\tilde{x}_1^{\frac{1}{2}}\left(1+\frac{2\tilde{x}_1^{\frac{3}{2}}}{i\zeta_{00}^3}J_2-\frac{\tilde{x}_1^3J_3}{i\zeta_{00}^6}\right)\Big/H'(\zeta_{00}), \tag{5.5}$$

$$\frac{\kappa_3}{\kappa_0} = \frac{3}{4U_0'^2}\,e^{\frac{1}{4}i\pi}\zeta_{00}^{\frac{1}{2}}\tilde{x}_1^{\frac{3}{2}}/H'(\zeta_{00}), \tag{5.6}$$

where $H(\zeta_{00})$ is defined by (4.52) and (5.2) and the primes on H denote derivatives with respect to ζ_{00}.

Since
$$\int_{\infty_1}^{\zeta_{00}} \mathrm{Ai}\,(\zeta)\,d\zeta$$
cannot become infinite as $\tilde{x}_1 \to 0$, it follows from (4.52) and (5.3) that either (i) $\zeta_{00} \to 0$ or (ii) that $\mathrm{Ai}'\,(\zeta_{00}) \to 0$ in this limit. But, expanding (5.2) and (5.3) for small $\tilde{x}_1$ and ζ_{00} shows that condition (i) can only occur if $\arg \kappa_0 = \frac{5}{8}\pi i$, which corresponds to an upstream-propagating wave, or if $\arg \kappa_0$ has increased by more than a factor of 2π from its value at the neutral curve, where (4.46) and (4.47) hold. In the latter case, the eigensolution would have to exhibit growth upstream of the neutral curve – which certainly cannot occur. Hence we must conclude that

$$\mathrm{Ai}'\,(\zeta_{00}) \to 0 \quad \text{as} \quad \tilde{x}_1 \to 0. \tag{5.7}$$

It therefore follows that
$$\zeta_{00} \to \zeta_n \tag{5.8}$$
for some $n = 1, 2, \ldots$, where ζ_n is determined by (3.13), which can be thought of as the characteristic equation for the asymptotic eigensolution (3.15). Thus the limiting form of the characteristic equation for the Orr–Sommerfeld equation coincides with the characteristic equation for the asymptotic eigensolutions.

Equation (5.3) has been solved numerically. The results are shown in figure 2.

On the lower branch of the neutral-stability curve the solution to (4.42) is given by (4.46) and (4.47) to lowest order of approximation in $\alpha = \epsilon\bar{\alpha}$ and $c = \epsilon\bar{c}$. These equations must therefore determine the neutrally stable solutions of (5.2) and (5.3) (i.e. the solutions corresponding to real κ_0) when α, c and R are expressed in terms of κ_0 and $\tilde{x}_1$.

Substituting (4.46) and (4.47) into (4.12) and (4.40), using (5.1), and noting that $\zeta_0 = \zeta_{00} + O(\epsilon)$, we find that

$$\zeta_{00} = (2{\cdot}296)\,(1{\cdot}002)^{\frac{1}{3}}\exp\left(-\tfrac{5}{6}\pi i\right) \quad \text{and} \quad \tilde{x}_1 \equiv 2x_1/(U_0')^2 = (2.296)^{\frac{4}{3}}\,(1{\cdot}002)^{\frac{2}{3}} \approx 3{\cdot}033$$

on the lower branch of the neutral-stability curve. Figure 2 shows that the curves

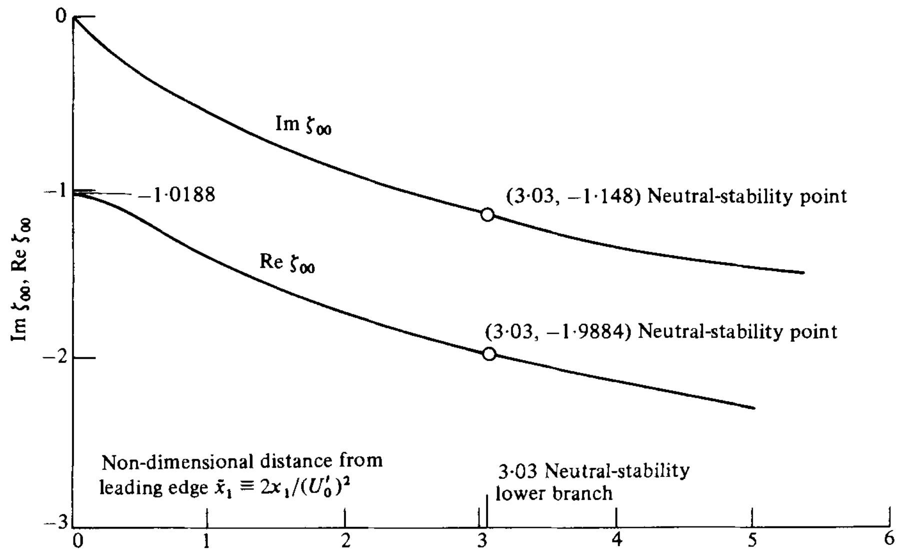

FIGURE 2. Variation of complex eigenvalue ζ_{00} with non-dimensional distance x_1.

pass through this point. It also shows that ζ_{00} approaches $-1{\cdot}0188$ as $\tilde{x}_1 \to 0$. As can be seen from the table on p. 478 of Abramowitz & Stegun (1964), this is the smallest root of (3.13), i.e. it is equal to ζ_1. This provides a numerical verification of our conclusion that (3.13) is the appropriate limiting form of the characteristic equation (5.3).

It now follows from (3.9), (4.3), (4.20), (4.51), (5.8) and (5.2) that

$$\kappa_0 \to i(2x_1)^{\frac{1}{2}}\lambda/U_0' = i\epsilon(2x)^{\frac{1}{2}}\lambda/U_0' \quad \text{as} \quad x_1 \to 0, \tag{5.9}$$

where λ is given by (3.9). Or more precisely, $\kappa_0 = i\epsilon\ (2x)^{\frac{1}{2}}\lambda/U_0' + 3C_0 x_1^2 + o(x_1^2)$ as $x_1 \to 0$, where C_0 is an $O(1)$ constant. On the other hand, it follows from (4.41), the right-hand side of (4.52), (5.7) and table 10.13 on p. 478 of Abramowitz & Stegun (1964) that both $H'(\zeta_{00})$ and $H'(\zeta_{00})/H''(\zeta_{00})$ are non-zero constants when $\tilde{x}_1 = 0$. Hence (5.4)–(5.6), (4.51) and (5.9) imply that

$$\kappa_1 \to \tfrac{5}{2}C_1 x_1^{\frac{3}{2}}, \quad \kappa_2 \to 2C_2 x_1, \quad \kappa_3 \to 3C_3 x_1^2 \tag{5.10}$$

as $x_1 \to 0$, where C_1, C_2 and C_3 are $O(1)$ constants. We therefore conclude from (4.3), (4.20), (5.1) and (5.9) that

$$\exp\left\{\frac{i}{\epsilon}\int_0^x \kappa(x_1,\epsilon)\,dx\right\} = \exp\left\{-\frac{\lambda(2x)^{\frac{3}{2}}}{3U_0'} + i\epsilon^3 x^2 (C_0 x + C_1 x^{\frac{1}{2}} + C_2) + iC_3\epsilon^6(\ln\epsilon)\, x^3\right\} + O(\epsilon^4)$$

$$= \exp\left[\frac{-\lambda(2x)^{\frac{3}{2}}}{3U_0'}\right][1 + i\epsilon^3 x^2 (C_0 x + C_1 x^{\frac{1}{2}} + C_2) + O(\epsilon^4)] \tag{5.11}$$

when $\epsilon \to 0$ and $x = O(1)$.

It now follows from (4.5) and (4.20) that *the exponential term in* (4.4) *does indeed match the exponential term in the asymptotic eigensolution* (3.15).

Equations (4.50) and (4.52) were solved numerically to determine κ as a function of x_1. The results, which are plotted in figure 3, show that Im κ is zero when x_1 is on the neutral-stability curve. The dashed curve is a plot of the real and imaginary parts of the right-hand side of (5.9) with λ given by (3.9) for $n = 1$. (Note that Re $i\lambda$ = Im $i\lambda$ = $0{\cdot}6876$ in this case.) The $\epsilon = 0$ curves represent κ_0, the lowest-order

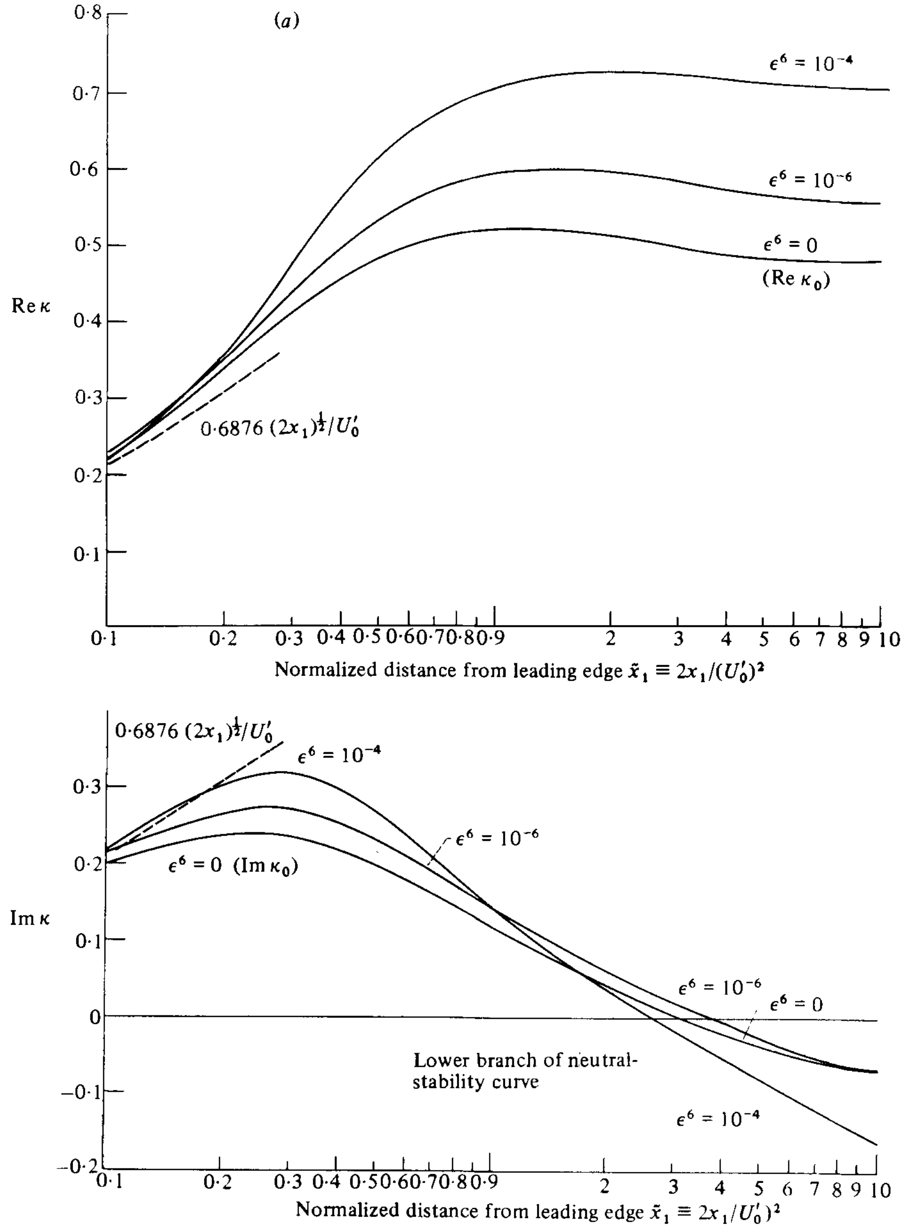

FIGURE 3. Variation of ϵ times the complex wavenumber with distance from the leading edge. (*a*) Real part; (*b*) imaginary part.

approximation to κ. Thus, the figure shows that (5.9) holds for the lowest-order eigenvalue and that $\kappa \to \kappa_0$ as $x_1 \to 0$ with $\epsilon > 0$.

5.2. *Matching of amplitudes*

We first consider the main inviscid region where $\eta = O(1)$. Inserting (4.23), (5.1), (5.9) and (5.10) into (4.28) shows that

$$\gamma \to U + \frac{iU_0'}{\lambda(2x)^{\frac{1}{2}}} + O(\epsilon^3) \quad \text{as} \quad x_1 \to 0.$$

Comparing this with the first line of (3.7) and using (4.11) shows that

$$\gamma \to g_0/(2x)^{\frac{1}{2}} \quad \text{as} \quad x_1 \to 0 \quad \text{with} \quad \eta = O(1). \tag{5.12}$$

Hence the amplitudes match to within a power of x, which can be accounted for when A and x^τ are matched.

We now consider the solution in the wall layer where $\bar{\eta} = \eta/\epsilon = O(1)$ and $\gamma = \epsilon\bar{\gamma}_0 + O(\epsilon^2)$.

It follows from (4.23), (4.50), (5.1), (5.2) and (5.8)–(5.10) that

$$\bar{c} \to U_0'/(i\epsilon\lambda(2x)^{\frac{1}{2}}), \quad \zeta_0 \to \zeta_n \quad \text{as} \quad x_1 \to 0. \tag{5.13}$$

Inserting these into (4.39) we find that $\zeta \to \zeta_b$ as $x_1 \to 0$ where ζ_b is defined in (3.11) with σ given by (3.12). Inserting this together with (5.13) into (B 1), introducing σ as a new variable of integration, and comparing the result with (3.7*b*) shows that (5.12) also holds in the inner viscous layer where $\bar{\eta} = O(1)$.

6. Discussion of results

The remnant of the unsteady boundary-layer solution oscillates about a Stokes-shear-layer type of solution with progressively decreasing amplitude. Mathematically, these oscillations are represented by asymptotic eigensolutions. In this paper, we consider only the asymptotic eigensolutions of Lam & Rott (1960), whose wavelength decreases with increasing distance downstream (it decreases like $x^{-\frac{1}{2}}$). This reduction in spatial scale gives rise to cross-stream inertia effects, which are absent in the unsteady boundary-layer region and which can eventually (i.e. when $x = O(\epsilon^{-2})$) destabilize the flow – causing it to behave like a spatially growing Tollmien–Schlichting wave.

The reduction in wavelength (and consequently in phase speed) allows free-stream disturbances to couple with Tollmien–Schlichting waves even when the wavelength of the former is very much larger than that of the latter. The reduction occurs because the asymptotic eigensolutions satisfy a homogeneous equation, which does not contain a free-stream pressure term to balance the temporal acceleration term. Since the latter term cannot be entirely balanced by viscous effects it must, in the main, be balanced by the convective acceleration term.

The temporal and convective acceleration terms would balance exactly if the phase Φ of the disturbance were $t - \int dx/U$.

Near the wall, $U \propto \eta \propto y/x^{\frac{1}{2}}$, so that $\Phi - t \propto x^{\frac{3}{2}}$. Thus the wavelength of this disturbance decreases like $x^{-\frac{1}{2}}$.

The wavelength of the asymptotic eigensolution (3.6) decreases like $x^{-\frac{1}{2}}$ for a similar reason, i.e. because it must penetrate into a region of decreasing mean velocity while producing no pressure fluctuations.

The Lam & Rott asymptotic eigensolutions match onto Tollmien–Schlichting waves far downstream in the flow. The characteristic equation (4.55), which determines the eigenvalues of these waves, has one root for each of the asymptotic eigensolutions of the unsteady boundary-layer equation. *Only the lowest-order asymptotic eigensolution of the unsteady boundary-layer equation turns into a spatially growing Tollmien–Schlichting wave.* The remaining eigensolutions match onto Tollmien–Schlichting waves that continue to decay.

Figure 3 shows that the imaginary part of the wavenumber κ of the former Tollmien–Schlichting wave decreases very rapidly with increasing downstream distance when $\tilde{x}_1 \equiv 2x_1/U_0'^2 > 0{\cdot}3$. In this way, the lowest-order asymptotic eigen-

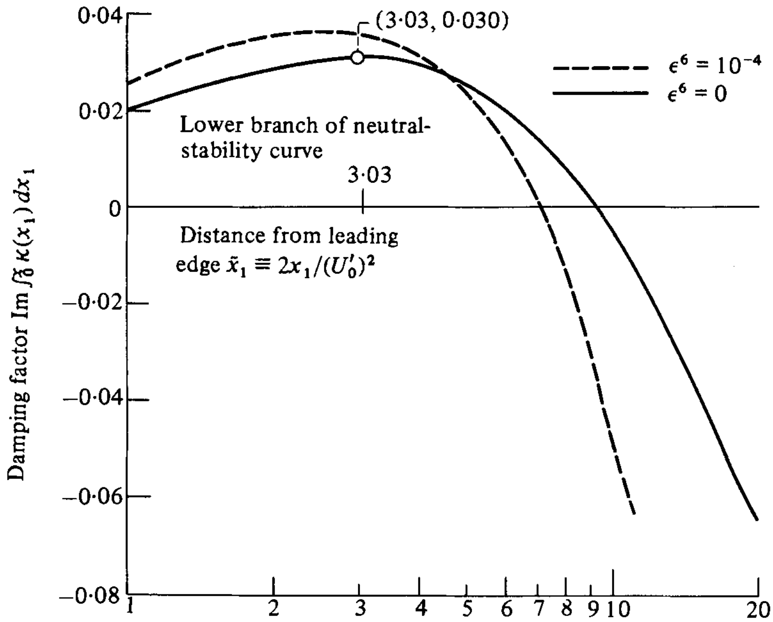

FIGURE 4. Variation of damping factor with streamwise distance.

solution, which is at first quite highly damped, eventually turns into a growing disturbance.

The figure also shows that the real part of the wavenumber (i.e. the reciprocal wavelength) at first increases with x_1, reaches a maximum when $\tilde{x}_1 = 2x_1/(U_0')^2$ is near unity, and remains relatively constant thereafter. Thus, the initial wavelength reduction, which occurs because of the absence of pressure fluctuations, ultimately produces the pressure fluctuations needed to keep the wavelength relatively constant. One might say that a quasi-equilibrium condition is reached when x_1 is $O(1)$. Our numerical solution of (5.2) and (5.3) shows that $\kappa_0/(2x_1)^{\frac{1}{2}}$ is relatively independent of x_1 for the Tollmien–Schlichting waves corresponding to the remaining asymptotic eigensolutions.

The real part of the exponent in (4.4) is a measure of the amount of damping the Tollmien–Schlichting wave undergoes. This quantity multiplied by $-\epsilon^3$ is, to the lowest order of approximation, equal to $\mathrm{Im}\int_0^x \kappa(x_1)\,dx_1$, which is plotted in figure 4 as a function of the normalized distance $\tilde{x}_1 = 2x_1/(U_0')^2$. It attains its maximum value at the neutral-stability point, which occurs at $\tilde{x}_1 \approx 3{\cdot}03$ when $\epsilon = 0$. Beyond this point it begins to decrease until it becomes negative, which indicates that the instability wave has grown beyond its initial upstream value. The maximum damping is roughly equal to $-3{\cdot}62$ when the characteristic Reynolds number ϵ^{-6} is 10^4.

It follows from (3.3), (4.4), (4.5), (4.13), (4.20) and the remarks at the end of the first paragraph of §5 that the streamwise velocity fluctuation associated with the Tollmien–Schlichting wave is given by

$$u_1 = \epsilon^{-2\tau}(2x_1)^{-\frac{1}{2}} A(x_1) \frac{\partial\gamma}{\partial\eta} \exp\left[\frac{i}{\epsilon}\int_0^x \kappa(x_1, \epsilon)\,dx\right], \tag{6.1}$$

where τ is given by (3.16) and, in the main part of the boundary layer (see (4.28)), $\partial\gamma/\partial\eta = U' + O(\epsilon)$. Since the slowly varying function $A(x_1)/(2x_1)^{\frac{1}{2}}$ is $O(1)$, the dominant Reynolds-number dependence of the amplitude of the Tollmien–Schlichting wave is given by the factor $\epsilon^{-2\tau}$ and since A is only determined to within an arbitrary constant, it follows from the first paragraph of §5 that it can be normalized so that

$$A(x_1)/(2x_1)^{\frac{1}{2}} \rightarrow x_1^{\tau} \quad \text{as} \quad x_1 \rightarrow 0. \tag{6.2}$$

Then the Tollmien–Schlichting wave (6.1) arising from any given free stream disturbance will be multiplied by the same $O(1)$ constant as the corresponding asymptotic eigensolution of the unsteady boundary-layer equation.

It is worth noting that the results of this paper are entirely independent of the nature of the free-stream velocity perturbation u_∞. More importantly, they also apply to any finite-thickness flat plate whose mean pressure gradient is sensibly zero in the downstream region where $x > 1$. The unsteady Blasius boundary-layer equation (3.2) still holds in this region, but its asymptotic solution now corresponds to a different 'upstream boundary condition'. This will no doubt have an important effect on the constants that multiply the asymptotic eigensolutions, but we have not attempted to calculate these here.

The author would like to thank Professors Eli Reshotko and Franklin K. Moore for their helpful comments during the course of this work. He would also like to thank Bruce Auer for carrying out the numerical computations.

Appendix A. Amplitude corrections for asymptotic eigensolutions

It was shown by Lam & Rott (1960) that (3.2) possesses the homogeneous solution (i.e. a solution with the free-stream forcing term h put equal to zero)

$$\psi_c = (2x)^{\frac{1}{2}} p(x) - \frac{(2x)^{-\frac{1}{2}}}{i} [2xp_x + p(x)] F'(\eta), \tag{A 1}$$

where $p(x)$ can be any differential function of x, and $F(\eta)$ is the Blasius function defined by (3.12)–(3.14).

Ackerberg & Phillips (1972) point out that (A 1) can be considered as an eigensolution for the outer flow since $\partial\psi_c/\partial\eta$ satisfies a homogeneous outer boundary condition. However, it does not satisfy the wall boundary conditions (3.5) but, as shown by Ackerberg & Phillips, it can be asymptotically matched onto an 'inner solution' that does satisfy these conditions.

This latter solution is obtained by introducing the new independent variables σ (which is defined in (3.12)) and

$$\alpha = \xi^{-1} = x^{-\frac{1}{2}} \tag{A 2}$$

into (3.2) with $h = 0$ to obtain

$$4\tilde{\psi}_{\sigma\sigma\sigma} + 2^{\frac{3}{2}}\alpha F\tilde{\psi}_{\sigma\sigma} + 4i\tilde{\psi}_\sigma - 2\alpha^2 F'(\sigma\tilde{\psi}_{\sigma\sigma} - \alpha\tilde{\psi}_{\sigma\alpha}) + \sqrt{2}\,\alpha^3 F''(\sigma\tilde{\psi}_\sigma - \alpha\tilde{\psi}_\alpha) = 0, \tag{A 3}$$

where we have put

$$\tilde{\psi}(\sigma, \alpha) \equiv \psi_c(\eta, x). \tag{A 4}$$

The variable σ is presumed to be of order one in the wall layer. Then it follows from (3.2) that η will be small, and we can approximate the Blasius function by the first two terms in its Taylor-series expansion:

$$F = \frac{U_0'}{2}\eta^2 - \frac{(U_0')^2\eta^5}{5!} + \dots \quad \text{as} \quad \eta \rightarrow 0. \tag{A 5}$$

Inserting this into (A 3) and dropping terms that are higher order than α^3 and $\alpha^7\partial\tilde{\psi}/\partial\alpha$, we obtain

$$4\tilde{\psi}_{\sigma\sigma\sigma}+4i\tilde{\psi}_{\sigma}+\sqrt{2}\,\alpha^4U_0'(\sigma\tilde{\psi}_{\alpha\sigma}-\tilde{\psi}_{\alpha})$$
$$=\tfrac{1}{2}U_0'\left[\sqrt{2}\,\alpha^3(\sigma^2\tilde{\psi}_{\sigma\sigma}-2\sigma\tilde{\psi}_{\sigma})+\frac{U_0'}{24}\alpha^7\sigma^3(\sigma\tilde{\psi}_{\sigma\alpha}-4\tilde{\psi}_{\alpha})\right]. \quad \text{(A 6)}$$

Inserting the expansion (3.15) into this result, we find upon equating coefficients of like powers of α that $L_p g_0 = 0$,

$$L_p g_1 = (U_0'/2^{\frac{5}{2}})\,[4\tau(\sigma g_0'-g_0)+\sigma^2 g_0''-2\sigma g_0'+\frac{\lambda}{12}(\sigma g_0'-4g_0)\,\sigma^3], \quad \text{(A 7)}$$

where we have put
$$L_p \equiv \frac{d^3}{d\sigma^3}+i\frac{d}{d\sigma}+\lambda\left(\sigma\frac{d}{d\sigma}-1\right) \quad \text{(A 8)}$$

and the primes now denote differentiation with respect to σ.

Equation (3.5) implies that g_0 and g_1 must satisfy the wall boundary conditions

$$g_0 = g_0' = 0 \quad \text{at} \quad \sigma = 0, \quad \text{(A 9)}$$
$$g_1 = g_1' = 0 \quad \text{at} \quad \sigma = 0. \quad \text{(A 10)}$$

Equation (A 6) is the same as Ackerberg & Phillips' equation (4.4) for the σ-dependent part of the inner solution. They showed that the solution to this equation that satisfies the wall boundary conditions (A 9) and does not grow exponentially fast as $\sigma \to 0$ (which is requied in order that it be able to match a solution of the form (A 1)) is given by (3.7*b*). This proves our contention that x^τ times the Ackerberg & Phillips solution satisfies the governing equations to within an error $O(\alpha^3)$ times that solution – at least in the inner region. We will complete the proof of this assertion when we show that matching with (A 1) requires p to be equal to x^τ times the p given by Ackerberg & Phillips plus terms that are smaller by a factor of α^3.

Ackerberg & Phillips show that

$$g_0 \sim U_0'\left(\frac{i}{\lambda}+\sigma\right)+\text{exponentially small terms} \quad \text{as} \quad \sigma\to\infty. \quad \text{(A 11)}$$

The right-hand side of (A 7) therefore behaves like

$$-(U_0')^2\left[\frac{4i\tau}{\lambda}+2\sigma+\frac{\lambda\sigma^3}{3}\left(\frac{i}{\lambda}+\frac{3\sigma}{4}\right)\right]2^{-\frac{5}{2}}+\text{exponentially small terms} \quad \text{as} \quad \sigma\to\infty.$$

It follows that
$$g_1 \sim (U_0')^2\left(\frac{4i\tau}{\lambda^2}-\frac{\sigma^4}{12}\right)2^{-\frac{5}{2}}+Kg_0, \quad \text{(A 12)}$$

where K is a constant.

On the other hand, inserting (A 2), (A 5) and (3.12) into (A 1), we find that for small η the outer solution behaves like

$$\psi_c \sim \frac{\sqrt{2}}{\alpha}p(\alpha)-\frac{\alpha}{\sqrt{2}\,i}(p-\alpha p_\alpha)\left[\frac{U_0'\alpha\sigma}{\sqrt{2}}-\frac{(U_0')^2(\alpha\sigma)^4}{4(4!)}+\ldots\right] \quad \text{as} \quad \eta\to 0.$$

Hence, inserting (A 11) and (A 12) into (3.15), we see that the inner and outer solutions will match if we put

$$p = p_0+\alpha^3 p_1, \quad \text{(A 13)}$$

where
$$p_0 = \frac{\alpha^{1-2\tau}}{\lambda\sqrt{2}}\,U_0' i\exp\left[\frac{-\lambda 2^{\frac{3}{2}}}{3\alpha^3 U_0'}\right], \quad \text{(A 14)}$$

$$p_1 = \left(K + \frac{\tau U_0'}{\sqrt{2}\,\lambda}\right) p_0. \tag{A 15}$$

Inserting this into (A 1) we see that the lowest-order solution is indeed equal to x^τ times Ackerberg & Phillips' outer solution given by (3.7 *a*).

In order to determine τ we multiply both sides of (A 7) by $dw/d\sigma$ and integrate the result from zero to infinity. But, since $w(\sigma)$ satisfies Airy's equation

$$\frac{d^2 w}{d\sigma^2} + (\lambda\sigma + i)\, w = 0, \tag{A 16}$$

we find upon integrating by parts and using (3.11), (3.13) and (A 10) that

$$\int_0^\infty \frac{dw}{d\sigma} L_p g_1 \, d\sigma = -\int_0^\infty \frac{d^2 w}{d\sigma^2}\frac{d^2 g_1}{d\sigma^2}\, d\sigma - \int_0^\infty (\lambda\sigma + i)\, w \frac{d^2 g_1}{d\sigma^2}\, d\sigma - [i g_1'(0) - \lambda g_1(0)]\, \mathrm{Ai}\,(\zeta_0)$$
$$= 0.$$

It now follows from (A 7) that τ is given by (3.16).

Appendix B. Calculation of wall-layer solution

Since $d^2\bar{\gamma}_0/d\zeta^2$ satisfies (4.41) it follows from (4.37) and (4.48) that

$$\bar{\gamma}_0 = a_3 \left[\int_\zeta^{\zeta_0} \xi \,\mathrm{Ai}\,(\xi)\, d\xi + \zeta \int_{\infty_1}^{\zeta} \mathrm{Ai}\,(\zeta)\, d\zeta - \zeta_0 \int_{\infty_1}^{\zeta_0} \mathrm{Ai}\,(\zeta)\, d\zeta + \frac{U_0' \bar{\eta} \zeta_0}{\bar{c}} \int_{\infty_1}^{\zeta_0} \mathrm{Ai}\,(\zeta)\, d\zeta \right],$$

where a_3 is an arbitrary function of x_1 and the subscript 1 on ∞ is used to indicate that the path of integration tends to infinity in the sector $-\frac{1}{3}\pi < \arg \zeta < \frac{1}{3}\pi$. Inserting (4.39) to eliminate $\bar{\eta}$, using the fact that $U = U_0'\eta + O(\eta^4)$, and matching with (4.28) yields

$$\bar{\gamma}_0 = \frac{\bar{c} \int_{\zeta_0}^{\zeta} (\zeta - \xi)\, \mathrm{Ai}\,(\xi)\, d\xi}{\zeta_0 \int_{\infty_1}^{\zeta_0} \mathrm{Ai}\,(\xi)\, d\xi}. \tag{B 1}$$

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1964 *Handbook of Mathematical Functions.* National Bureau of Standards Appl. Math. Ser. 55.

ACKERBERG, R. C. & PHILLIPS, J. H. 1972 *J. Fluid Mech.* **51**, 137.

BROWN, S. N. & STEWARTSON, K. 1973 *Proc. Camb. Phil. Soc.* **73**, 493.

COLE, J. D. 1968 *Perturbation Methods in Applied Mathematics*, p. 11. Blaisdell.

GASTER, M. 1974 *J. Fluid Mech.* **66**, 465.

GOLDSTEIN, M. E. 1982 *NASA* TM 83026.

GOLDSTEIN, S. 1960 *Lectures on Fluid Mechanics.* Interscience.

LAM, S. H. & ROTT, N. 1960 Theory of linearized time-dependent boundary layers. *Cornell Univ. Grad. School of Aero. Engng Rep.* AFOSR TN-60-1100.

LIGHTHILL, M. J. 1954 *Proc. R. Soc. Lond.* A **224**, 1.

LIN, C. C. 1945 *Q. Appl. Math.* **3**, 117.

LIN, C. C. 1946 *Q. Appl. Math.* **3**, 277.

LIN, C. C. 1955 *The Theory of Hydrodynamic Stability.* Cambridge University Press.

MESSITER, A. F. 1970 *SIAM J. Appl. Math.* **18**, 241.

MILES, J. W. 1962 *J. Fluid Mech.* **13**, 427.

MOORE, F. K. 1951 *NACA TN* 2471.

MORKOVIN, M. V. 1969 *Air Force Flight Dyn. Lab., Wright–Patterson AFB, Ohio, Rep.* AFFDL-TR-68-149.

NAYFEH, A. H. 1973 *Perturbation Methods*. Wiley.

REID, W. H. 1965 In *Basic Developments in Fluid Dynamics*, vol. I (ed. Maurice Holt), pp. 249–307. Academic.

RESHOTKO, E. 1976 *Ann. Rev. Fluid Mech.* **8**, 311.

SARIC, W. S. & NAYFEH, A. H. 1975 *Phys. Fluids* **18**, 945.

SMITH, F. T. 1979 *Proc. R. Soc. Lond* A **366**, 91.

STEWARTSON, K. 1969 *Mathematika* **16**, 106.

TOLLMIEN, W. 1929 *Nachr. Ges. Wiss. Göttingen, Math.-Phys. Kl.* **21**. [Translated as *NACA* 609 (1931).]

TOLLMIEN, W. 1947 *Z. angew. Math. Mech.* **27**, 33, 70.

VAN DYKE, M. 1975 *Perturbation Methods in Fluid Mechanics*, annotated edn. Parabolic.